A trigonometric inequality

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Let $a,b \in (0,\pi/4)$, prove that

$$\frac{\sin^n a+\sin^n b}{(\sin a+\sin b)^n} \geq \frac{\sin^n 2a+\sin^n 2b}{(\sin 2a+\sin 2b)^n},$$ for all $n \in \mathbb{N}$.

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Assume due to symmetry with respect to a,b that $a \geq b$ and denote

$t := \dfrac{\sin a}{\sin b}, p := \dfrac{\sin 2a}{\sin 2b}$.

Since $0 < b \leq a < \pi/4$ and $\sin x$ increase in $[0,\pi/2]$ we obtain that $t,p \geq 1$.

Also note that $t \geq p$ because $0 < b \leq a < \pi/4$ implies $\dfrac{\sin a}{\sin b} \geq \dfrac{\sin 2a}{\sin 2b} \Leftrightarrow$ $1 \geq \dfrac{\cos a}{\cos b} \Leftrightarrow \cos b \geq \cos a$. Since $\dfrac{\sin^n a+\sin^n b}{(\sin a+\sin b)^n} \geq \dfrac{\sin^n 2a+\sin^n 2b}{(\sin 2a+\sin 2b)^n} \Leftrightarrow$

$$\frac{\left(\frac{\sin a}{\sin b}\right)^n+1}{\left(\frac{\sin a}{\sin b}+1\right)^n} \geq \frac{\left(\frac{\sin 2a}{\sin 2b}\right)^n+1}{\left(\frac{\sin 2a}{\sin 2b}+1\right)^n} \Leftrightarrow \frac{t^n+1}{(t+1)^n} \geq \frac{p^n+1}{(p+1)^n}$$ suffices to prove

that latter inequality holds for any $t \geq p \geq 1$.

We have $(t^n+1)(p+1)^n-(t+1)^n(p^n+1) = (p+1)^n-(t+1)^n+t^n(p+1)^n-p^n(t+1)^n =$

$(t(p+1)-p(t+1))\sum_{k=0}^{n-1}(t(p+1))^k(p(t+1))^{n-1-k}-(t-p)\sum_{k=0}^{n-1}(t+1)^{n-1-k}(p+1)^k =$

$(t-p)\sum_{k=0}^{n-1}\left(t^k(p+1)^k p^{n-1-k}(t+1)^{n-1-k}-(t+1)^{n-1-k}(p+1)^k\right) =$

$(t-p)\sum_{k=0}^{n-1}(t+1)^{n-1-k}(p+1)^k(t^k p^{n-1-k}-1) \geq 0.$